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Linear independence of intertwining operators

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Abstract

For an irreducible admissible representation of a connected reductive p-adic group, we consider standard intertwining operators holomorphic at zero. Using algebraic methods connected with the structure of linear algebraic groups, we control supports of particularly chosen functions from the induced space. We prove linear independence of standard intertwining operators. This is used to extend the definition of the R-group from a square integrable representation to its Aubert involution

1. Introduction

Intertwining operators are important for the trace formula and for understanding reducibility of induced representations. Some properties of intertwining operators are condensed in R-groups. Classically, R-groups are defined for square-integrable representations, using the Plancherel measure.

Let G be a connected reductive split p-adic group, and M a standard Levi subgroup of G. Let σ be an irreducible square-integrable representation of M. The R-group is a subgroup of the Weyl group. Attached to each $r \in R$ is a self intertwining operator $A(\sigma, r)$ of the induced representation $i_{G,M}(\sigma)$. The set $\{A(\sigma, r) \mid r \in R\}$ is a basis for the commuting algebra

$$C(\sigma) = \mathrm{Hom}_G\big(i_{G,M}(\sigma), i_{G,M}(\sigma)\big).$$

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The R-group could also be defined in terms of the L-group and the Langlands correspondence. In this context, Arthur proposed a conjectural description of the R-group for a nontempered unitary representation.

Some nontempered unitary representations are duals, in the sense of [3] or [18], of square-integrable representations. Let $\hat{\sigma}$ denote the Aubert involution of the square-integrable representation σ. We assume that $\hat{\sigma}$ is unitary. In [4], it is proved that

$$C(\sigma) \cong C(\hat{\sigma}).$$

[4] also establishes the connection between standard intertwining operators for σ and $\hat{\sigma}$. This is done under the hypothesis that the cocycle η is trivial and the same hypothesis we need in Sections 5 and 6. In this paper, we prove that the set of intertwining operators $\{A(\hat{\sigma}, r) \mid r \in R\}$ is linearly independent and therefore is a basis for the commuting algebra $C(\hat{\sigma})$ (Theorem 5.1). In this manner, we define the R-group for a class of nontempered unitary representations (representations whose Aubert involution is square integrable).

The linear independence of the operators $A(\hat{\sigma}, r)$, $r \in R$, follows from Theorem 4.3, which concerns any irreducible admissible representation. It states that standard intertwining operators holomorphic at zero are linearly independent and the proof is based on the structure of reductive groups. Using the Bruhat decomposition and the structure of G, we control supports of particularly chosen functions from the induced space (Lemma 4.2) and this enables us to prove linear independence. Theorem 4.3 could play a role in extending the definition of the R-group to other classes of nontempered unitary representations.

We now give a short summary of the paper. In Section 2, we introduce notation. Section 3 is about the structure of reductive groups. The linear independence of intertwining operators is proved in Section 4. In Section 5, we define the R-group for $\hat{\sigma}$ by proving that σ and $\hat{\sigma}$ have the same R-group. In Section 6, we describe explicitly the action of normalized operators on irreducible subspaces of $i_{G,M}(\sigma)$ and express the result using a trace formulation (Theorem 6.1).

2. Notation

Let F be a p-adic field. Let G be the group of F-points of a connected, split, reductive algebraic group $\mathbf{G}$ defined over F. Fix a Borel subgroup B and a maximal split torus $T \subset B$. Let $U_\emptyset$ be the unipotent radical of B. Then $B = TU_\emptyset$.

Denote by W the Weyl group of G with respect to T. Let $\mathcal{O}$ denote the ring of integers of F. We fix a set of representatives for W in $G(\mathcal{O})$ and by abuse of notation, use w to denote both the element of W and its representative.

Let Σ be the set of roots of G with respect to T. The choice of B determines a basis Δ of Σ (which consists of simple roots). It also determines the set of positive roots Σ^+ and the set of negative roots Σ^-.

Suppose that $P = MU$ is the standard parabolic subgroup corresponding to a set of simple roots $\Theta \subset \Delta$. Denote by P^- the opposite parabolic subgroup of P, i.e., the unique parabolic subgroup intersecting P in M. Let U^- be the unipotent radical of P^-. Define

$$U_w = U_\emptyset \cap wU^- w^{-1}, \qquad U_w^- = w^{-1}U_w w = w^{-1}U_\emptyset w \cap U^-.$$

Denote by $W_\Theta = W(M/T)$ the Weyl group of M with respect to T. The set

$$[W_\Theta \backslash W / W_\Theta] = \{w \in W \mid w\Theta > 0,\ w^{-1}\Theta > 0\}$$

is a set of representatives of double cosets $W_\Theta \backslash W / W_\Theta$ [8,22]. As in [8], we define a partial ordering on $[W_\Theta \backslash W / W_\Theta]$: $x \leqslant y$ if PxP is contained in the closure of PyP. For each $w \in [W_\Theta \backslash W / W_\Theta]$, let

$$G_w = \bigcup_{x \geqslant w} PxP.$$

The set G_w is open in G and $x \geqslant y$ implies $G_x \subseteq G_y$.

Let (σ, V) be a smooth representation of M. Denote by $i_{G,M}(V)$ the set of all functions $f: G \to V$ satisfying [5,23]

(1) $f(umg) = \delta_P^{1/2}(m)\sigma(m)f(g)$, for all $u \in U$, $m \in M$, $g \in G$. (Here δ_P denotes the modular function of P.)
(2) There exists an open subgroup $K \subset G$ such that $f(gk) = f(g)$, for $g \in G$, $k \in K$.

Then $i_{G,M}(\sigma)$ is the representation of G on $i_{G,M}(V)$ defined by

$$\big(i_{G,M}(\sigma)(g)f\big)(x) = f(xg), \quad f \in i_{G,M}(V),\ x, g \in G$$

(G acts on $i_{G,M}(V)$ by right translations). We will also denote $i_{G,M}(V)$ by $i_{G,M}(\sigma)$. Set

$$W(\Theta) = \{w \in W \mid w(\Theta) = \Theta\}, \qquad W(\sigma) = \{w \in W(\Theta) \mid w\sigma \cong \sigma\},$$

where $w\sigma$ is defined in the usual way: $w\sigma(m) = \sigma(w^{-1}mw)$, $m \in M$.

Let A be the split component of M. Denote by $X(M)_F$ and $X(A)_F$ respectively the group of all F-rational characters of M and A. Let

$$\mathfrak{a} = \mathrm{Hom}\big(X(M)_F, \mathbb{R}\big) = \mathrm{Hom}\big(X(A)_F, \mathbb{R}\big)$$

be the real Lie algebra of A and

$$\mathfrak{a}^* = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R} = X(A)_F \otimes_{\mathbb{Z}} \mathbb{R}$$

its dual. Set $\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* \otimes \mathbb{C}$. There is a homomorphism (cf. [11]) $H_M: M \to \mathfrak{a}$ such that $q^{\langle \chi, H_M(m)\rangle} = |\chi(m)|$ for all $m \in M$, $\chi \in X(M)_F$. Given $\nu \in \mathfrak{a}^*$, let us write

$$\exp \nu = q^{\langle \nu, H_M(\cdot)\rangle}$$

for the corresponding character.

Suppose that σ is an irreducible admissible representation of M and $w \in W$ such that $w(\Theta) \subset \Delta$. The standard intertwining operator $\mathbf{A}(\nu, \sigma, w)$ is formally defined by

$$\mathbf{A}(\nu, \sigma, w) f(g) = \int_{U_w} f\left(w^{-1} u g\right) \mathrm{d}u,$$

where $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, $f \in i_{G,M}(\exp \nu \otimes \sigma)$, and $g \in G$ [21]. It converges absolutely for the real part of ν in a certain chamber [2] and

$$\nu \mapsto \mathbf{A}(\nu, \sigma, w)$$

has analytic continuation as meromorphic function of $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. If $\mathbf{A}(\nu, \sigma, w)$ is holomorphic at $\nu = 0$, we denote $\mathbf{A}(0, \sigma, w)$ by $\mathbf{A}(\sigma, w)$. We define a normalized intertwining operator

$$A'(\nu, \sigma, w) = n(\nu, \sigma, w)\mathbf{A}(\nu, \sigma, w),$$

where $n(\nu, \sigma, w)$ is a normalizing factor [2,17,21]. For $w \in W(\sigma)$, let $T_w : V \to V$ be an isomorphism between $w\sigma$ and σ [9,10]. For simplicity of notation, we will denote the induced operator $i_{G,M}(T_w)$ by T_w. Its action on $f \in i_{G,M}(V)$ is given by $(T_w f)(g) = T_w(f(g))$, $g \in G$ [5]. Define

$$A(\sigma, w) = T_w A'(\sigma, w).$$

This is an isomorphism between $i_{G,M}(\sigma)$ and $i_{G,M}(\sigma)$. We have

$$A(\sigma, w_2 w_1) = \eta(w_2, w_1) A(w_1\sigma, w_2) A(\sigma, w_1),$$

where $\eta(w_2, w_1)$ is given by $T_{w_2 w_1} = \eta(w_2, w_1) T_{w_2} T_{w_1}$.

3. Reductive groups

In this section, we prove some results concerning the Bruhat decomposition (Lemma 3.1) and the structure of G (Lemmas 3.3, 3.5, and Corollary 3.6). These results are used in Section 4 for computing standard intertwining operators.

Lemma 3.1.

(1) *Let $w_1, w_2 \in W$. Then $w_1 B w_2 B \cap B \neq \emptyset$ if and only if $w_1^{-1} = w_2$.*
(2) *Let $\Theta \subset \Delta$ and $P = P_\Theta$. Let $w_1, w_2 \in [W_\Theta \backslash W / W_\Theta]$. Then $w_1 P w_2 P \cap P \neq \emptyset$ if and only if $w_1^{-1} = w_2$.*

Proof. (1) According to the Bruhat decomposition ([6, Theorem 14.12], [7, Theorem 1]), G is the disjoint union of double cosets BwB, $(w \in B)$, i.e.,

$$G = \bigcup_{w \in W} BwB.$$

If $w_1 B w_2 B \cap B \neq \emptyset$, then $Bw_2B \cap w_1^{-1}B \neq \emptyset$. Note that $w_1^{-1}B$ is a subset of the double coset $Bw_1^{-1}B$. It follows that $w_1^{-1} = w_2$.

(2) Similar to (1), using the disjoint union decomposition

$$G = \bigcup_{w \in [W_\Theta \backslash W / W_\Theta]} PwP,$$

[8, Proposition 1.3.1]. □

For $\alpha \in \Sigma$, let $\mathbf{U}_\alpha$ be the corresponding root group [6, Theorem 13.18].

Lemma 3.2 [6, 14.4]. *Let* $\mathbf{H}$ *be a* $\mathbf{T}$*-stable closed subgroup of* $\mathbf{U}$. *Let*

$$\Sigma(\mathbf{H}) = \{\alpha \in \Sigma \mid \mathfrak{u}_\alpha \subset \mathfrak{h}\},$$

where $\mathfrak{u}_\alpha$ *and* $\mathfrak{h}$ *respectively are Lie algebras of* $\mathbf{U}_\alpha$ *and* $\mathbf{H}$. *Let* $\alpha_1, \ldots, \alpha_k$ *be the elements of* $\Sigma(\mathbf{H})$, *in any fixed order. Then*

$$\mathbf{U}_{\alpha_1} \times \cdots \times \mathbf{U}_{\alpha_k} \longrightarrow \mathbf{H}$$

is an isomorphism of varieties.

Let $\Theta \subset \Delta$ be the set of simple roots corresponding to $\mathbf{P} = \mathbf{MU}$. We define Σ_Θ to be the subset of roots in the linear span of Θ. Then

$$\Sigma_\Theta = \Sigma_\Theta^+ \cup \Sigma_\Theta^-,$$

where $\Sigma_\Theta^+ = \Sigma^+ \cap \Sigma_\Theta$, $\Sigma_\Theta^- = \Sigma^- \cap \Sigma_\Theta$. We have

$$\mathbf{U} = \prod_{\alpha \in \Sigma^+ - \Sigma_\Theta^+} \mathbf{U}_\alpha, \qquad \mathbf{U}_\Theta^- = \prod_{\alpha \in \Sigma^- - \Sigma_\Theta^-} \mathbf{U}_\alpha.$$

Lemma 3.3. *Let* $\alpha_1, \ldots, \alpha_k$ *be the elements of* $\Sigma^+ - \Sigma_\Theta^+$ *in any fixed order and* $\alpha_{k+1}, \ldots, \alpha_n$ *the elements of* Σ_Θ^+. *Let*

$$\widehat{P} = U_{\alpha_1} \cdots U_{\alpha_n} T U_{-\alpha_n} \cdots U_{-\alpha_{k+1}}.$$

Then $\widehat{P}U^- = BU_\emptyset^-$ is an open subset of G and the multiplication

$$m : \widehat{P} \times U^- \longrightarrow \widehat{P}U^-$$

is an isomorphism of F-varieties.

Proof. By Lemma 3.2, we have the following isomorphisms of varieties:

$$\mathbf{U}_{\alpha_1} \times \cdots \times \mathbf{U}_{\alpha_n} \longrightarrow \mathbf{U}_\emptyset, \tag{1}$$

$$\mathbf{U}_{-\alpha_1} \times \cdots \times \mathbf{U}_{-\alpha_n} \longrightarrow \mathbf{U}_\emptyset^-, \tag{2}$$

$$\mathbf{U}_{\alpha_1} \times \cdots \times \mathbf{U}_{\alpha_k} \longrightarrow \mathbf{U}, \tag{3}$$

$$\mathbf{U}_{-\alpha_1} \times \cdots \times \mathbf{U}_{-\alpha_k} \longrightarrow \mathbf{U}^-, \tag{4}$$

$$\mathbf{U}_{\alpha_{k+1}} \times \cdots \times \mathbf{U}_{\alpha_n} \longrightarrow \mathbf{U}_{\alpha_{k+1}} \cdots \mathbf{U}_{\alpha_n}. \tag{5}$$

According to [6, Corollary 14.14],

$$\mathbf{B} \times \mathbf{U}_\emptyset^- \longrightarrow \mathbf{B}\mathbf{U}_\emptyset^- \tag{6}$$

is an isomorphism of varieties and $\mathbf{B}\mathbf{U}_\emptyset^-$ is an open subset of G. Let

$$\mathbf{B}_M = \mathbf{T}\mathbf{U}_{\alpha_{k+1}} \cdots \mathbf{U}_{\alpha_n}, \qquad \widehat{\mathbf{M}} = \mathbf{B}_M \mathbf{U}_{-\alpha_{k+1}} \cdots \mathbf{U}_{-\alpha_n}, \qquad \widehat{\mathbf{P}} = \mathbf{U}\widehat{\mathbf{M}}.$$

Then

$$\mathbf{B}_M \times \mathbf{U}_{-\alpha_{k+1}} \cdots \mathbf{U}_{-\alpha_n} \longrightarrow \widehat{\mathbf{M}} \tag{7}$$

is an isomorphism of varieties [6, Corollary 14.14]. According to [6, 10.6],

$$\mathbf{U}_\emptyset \times \mathbf{T} \longrightarrow \mathbf{B} \tag{8}$$

and

$$\mathbf{U}_{\alpha_{k+1}} \cdots \mathbf{U}_{\alpha_n} \times \mathbf{T} \longrightarrow \mathbf{B}_M \tag{9}$$

are isomorphisms of varieties.

We have the following series of isomorphisms:

$$\begin{tikzcd}
\mathbf{B}\mathbf{U}_{\emptyset}^{-} \arrow[d, "(6)"] \\
\mathbf{B} \times \mathbf{U}_{\emptyset}^{-} \arrow[d, "(8)"] \\
\mathbf{U}_{\emptyset} \times \mathbf{T} \times \mathbf{U}_{\emptyset}^{-} \arrow[d, "{(1),(2)}"] \\
\mathbf{U}_{\alpha_1} \times \cdots \times \mathbf{U}_{\alpha_k} \times \mathbf{U}_{\alpha_{k+1}} \times \cdots \times \mathbf{U}_{\alpha_n} \times \mathbf{T} \times \mathbf{U}_{-\alpha_n} \times \cdots \times \mathbf{U}_{-\alpha_{k+1}} \times \mathbf{U}_{-\alpha_k} \times \cdots \times \mathbf{U}_{-\alpha_1} \arrow[d, "{(3),(5),(9),(4)}"] \\
\mathbf{U} \times \mathbf{B}_M \times \mathbf{U}_{-\alpha_n} \times \cdots \times \mathbf{U}_{-\alpha_{k+1}} \times \mathbf{U}^{-} \arrow[d, "{(5),(7)}"] \\
\mathbf{U} \times \widehat{\mathbf{M}} \times \mathbf{U}^{-}.
\end{tikzcd}$$

It follows that

$$\mathbf{U} \times \widehat{\mathbf{M}} \times \mathbf{U}^{-} \longrightarrow \mathbf{U}\widehat{\mathbf{M}}\mathbf{U}^{-} = \mathbf{B}\mathbf{U}_{\emptyset}^{-} \tag{10}$$

is an isomorphism. By restricting it to $\mathbf{U} \times \widehat{\mathbf{M}} \times \{1\}$, we obtain the isomorphism

$$\mathbf{U} \times \widehat{\mathbf{M}} \longrightarrow \mathbf{U}\widehat{\mathbf{M}} = \widehat{\mathbf{P}}. \tag{11}$$

Now, (10) and (11) give the isomorphism

$$\widehat{\mathbf{P}} \times \mathbf{U}^{-} \longrightarrow \mathbf{U} \times \widehat{\mathbf{M}} \times \mathbf{U}^{-} \longrightarrow \mathbf{U}\widehat{\mathbf{M}}\mathbf{U}^{-} = \mathbf{B}\mathbf{U}_{\emptyset}^{-} = \widehat{\mathbf{P}}\mathbf{U}^{-}. \qquad \square$$

The following lemma was communicated to us by Donu Arapura.

Lemma 3.4. *Let X and Y be isomorphic quasiprojective F-varieties. Then the topological spaces X and Y, with the locally compact topology, are homeomorphic.*

Lemma 3.5. *The multiplication*

$$m : P \times U^{-} \longrightarrow PU^{-}$$

is a homeomorphism with respect to the locally compact topology.

Proof. According to Lemmas 3.3 and 3.4,

$$m : \widehat{P} \times U^- \longrightarrow \widehat{P}U^-$$

is a homeomorphism with respect to the locally compact topology.

The multiplication is continuous. Since $P \cap U^- = \{1\}$, we have that the multiplication $m : P \times U^- \to PU^-$ is bijective. We will prove that m is open.

Let $\mathcal{V}$ be an open set in $P \times U^-$. We have to prove that $m(\mathcal{V})$ is open in PU^-. Take $p\bar{u} \in m(\mathcal{V})$. We will prove that there exists an open subset $\mathcal{W} \in PU^-$ such that $p\bar{u} \in \mathcal{W} \subset m(\mathcal{V})$. The set

$$(p, \bar{u})\big(\widehat{P} \times U^-\big) = p\widehat{P} \times \bar{u}U^- = p\widehat{P} \times U^-$$

is open in $P \times U^-$ and so is

$$\big(p\widehat{P} \times U^-\big) \cap \mathcal{V}.$$

Hence, there exist sets P_1 open in P and U_1^- open in U^- so that

$$(p, \bar{u}) \in pP_1 \times U_1^- \subset \big(\big(p\widehat{P} \times U^-\big) \cap \mathcal{V}\big).$$

Note that $1 \in P_1$, $\bar{u} \in U_1^-$. Let $\mathcal{W} = pP_1U_1^-$. Then $p\bar{u} \in \mathcal{W} \subset m(\mathcal{V})$. Since $m : \widehat{P} \times U^- \to \widehat{P}U^-$ is a homeomorphism, the set $P_1U_1^-$ is open in $\widehat{P}U^-$. It follows that $P_1U_1^-$ is open in PU^- and $\mathcal{W} = pP_1U_1^-$ is open in PU^-. □

Corollary 3.6. *The multiplication*

$$m : P \times U_w^- \longrightarrow PU_w^-$$

is a homeomorphism with respect to the locally compact topology.

4. Linear independence of intertwining operators

In this section, we consider supports of particularly chosen functions from the induced space, under the action of standard intertwining operators. This is used for proving that standard intertwining operators holomorphic at zero are linearly independent (Theorem 4.3).

Lemma 4.1. *Let K be a compact group and K_1 a compact open subgroup of K. Then there exists a compact open subgroup $K_0 \subset K_1$ which is normal in K.*

Proof. Set

$$K_0 = \bigcap_{k \in K} kK_1k^{-1}.$$

Then K_0 is normal in K. Let $h, k \in K$. Suppose that $hK_1 = kK_1$. Then there exists $k_1 \in K_1$ such that $h = kk_1$ and we have

$$hK_1h^{-1} = kk_1K_1k_1^{-1}k^{-1} = kK_1k^{-1}.$$

It follows that

$$K_0 = \bigcap_{i=1}^{n} k_i K_1 k_i^{-1},$$

where $\{k_1, \ldots, k_n\}$ is a set of representatives of K/K_1. (The set is finite because K is compact and K_1 is open.) It follows that K_0 is open. $\square$

Lemma 4.2. *Let (σ, V) be an irreducible admissible representation of M and $w \in W(\Theta)$, $w \neq 1$. Then there exists a function $f \in i_{G,M}(V)$ satisfying*:

(1) $f(1) = 0$.
(2) *The integral defining $\mathbf{A}(\sigma, w) f(1)$ is absolutely convergent and*

$$\mathbf{A}(\sigma, w) f(1) \neq 0.$$

(3) *For any $w_1 \in W(\Theta)$ such that $w_1 \ngeq w$, we have*

$$\mathbf{A}(\sigma, w_1) f(1) = 0.$$

Proof. Fix a non-zero vector $v \in V$. Let K be a compact open subgroup of G such that $\delta^{1/2}(k)\sigma(k)v = v$, for all $k \in K \cap M$. According to Lemma 4.1, we may assume that K is normal in $G(\mathcal{O})$.

Let $\mathcal{M} = m(P \times U_w^-) = PU_w^-$. The subgroup $U_w^- \cap K$ is open in U_w^- and

$$\mathcal{M}_0 = m\big(P \times (U_w^- \cap K)\big) = P(U_w^- \cap K) \subset \mathcal{M}$$

is open in $\mathcal{M}$ (Corollary 3.6). Hence, there exists a compact open subgroup $K_0 \subset G$ such that $K_0 \cap \mathcal{M} \subset \mathcal{M}_0$. We may assume that $w^{-1}K_0 \subset G_{w^{-1}}$ and that K_0 is normal in $G(\mathcal{O})$. In addition, we require K_0 to be a subgroup of K.

For any $\bar{u} = pk_0 \in U_w^- \cap PK_0$, we have

$$p^{-1}\bar{u} = k_0 \in K_0 \cap \mathcal{M} \subset \mathcal{M}_0,$$

so $\bar{u} \in U_w^- \cap K$. Therefore, $U_w^- \cap PK_0 \subset U_w^- \cap K$.

Define $f : G \to V$ by

$$f(g) = \begin{cases} \delta^{1/2}(m)\sigma(m)v & \text{if } g = muw^{-1}k \in Pw^{-1}K_0, \\ 0 & \text{otherwise.} \end{cases}$$

We prove that f is well-defined. Suppose that $p_1w^{-1}k_1 = p_2w^{-1}k_2$, for $p_1, p_2 \in P$, $k_1, k_2 \in K_0$. Then

$$p_2^{-1}p_1 = w^{-1}k_2k_1^{-1}w.$$

Since K_0 is a normal subgroup of $G(\mathcal{O})$, we have $p_2^{-1}p_1 = k \in K_0 \subset K$, so

$$\delta^{1/2}\big(p_2^{-1}p_1\big)\sigma\big(p_2^{-1}p_1\big)v = \delta^{1/2}\big(p_2^{-1}\big)\sigma\big(p_2^{-1}\big)\delta^{1/2}(p_1)\sigma(p_1)v = \delta^{1/2}(k)\sigma(k)v = v.$$

It follows that

$$\delta^{1/2}(p_1)\sigma(p_1)v = \delta^{1/2}(p_2)\sigma(p_2)v$$

and therefore f is well-defined. Further, f is smooth and satisfies $f(mug) = \delta^{1/2}(m) \times \sigma(m)f(g)$, for $m \in M$, $u \in U$, $g \in G$, so $f \in i_{G,M}(V)$.

We have

$$\operatorname{supp} f \subset Pw^{-1}K_0 \subset PG_{w^{-1}} = G_{w^{-1}} = \bigcup_{x \geqslant w^{-1}} PxP$$

and $\operatorname{supp} f \cap P = \emptyset$. Now,

$$\mathbf{A}(\sigma, w)f(1) = \int_{U_w} f\big(w^{-1}u\big)\,\mathrm{d}u = \int_{U_w} f\big(w^{-1}uww^{-1}\big)\,\mathrm{d}u = \int_{U_w^-} f\big(\bar{u}w^{-1}\big)\,\mathrm{d}\bar{u}.$$

By the definition of f, $f(\bar{u}w^{-1})$ is non-zero if $\bar{u}w^{-1} \in Pw^{-1}K_0$, or $\bar{u} \in Pw^{-1}K_0w = PK_0$. Therefore,

$$\mathbf{A}(\sigma, w)f(1) = \int_{U_w^- \cap PK_0} f\big(\bar{u}w^{-1}\big)\,\mathrm{d}\bar{u}.$$

Let $\bar{u} \in U_w^- \cap PK_0 \subset K$. Then $\bar{u} = pk_0 \in K$, where $p \in P$ and $k_0 \in K_0$. We have $p = \bar{u}k_0^{-1} \in K$, so

$$f\big(\bar{u}w^{-1}\big) = f\big(pk_0w^{-1}\big) = \delta^{1/2}(p)\sigma(p)v = v.$$

Therefore,

$$\mathbf{A}(\sigma, w)f(1) = \int_{U_w^- \cap PK_0} v\,\mathrm{d}\bar{u} = \operatorname{meas}\big(U_w^- \cap PK_0\big)v.$$

Since $U_w^- \cap PK_0 \subset K$, the set $U_w^- \cap PK_0$ has finite measure and the integral is absolutely convergent. On the other hand, $U_w^- \cap K_0 \subset U_w^- \cap PK_0$ implies that $\operatorname{meas}(U_w^- \cap PK_0) \neq 0$. It follows that $\mathbf{A}(\sigma, w)f(1) \neq 0$.

For (3), let $w_1 \ngeqslant w$. Then

$$\mathbf{A}(\sigma, w_1) f(1) = \int\limits_{U_{w_1}} f\bigl(w_1^{-1}u\bigr)\,\mathrm{d}u.$$

Suppose that $f(w_1^{-1}u) \neq 0$, for some $u \in U_{w_1}$. Then $w_1^{-1}u \in \operatorname{supp} f \subset G_{w^{-1}}$ and

$$u \in U \cap w_1 G_{w^{-1}} = U \cap w_1 \biggl(\bigcup_{x \geqslant w^{-1}} PxP\biggr),$$

which is empty, according to Lemma 3.1. Therefore, $f(w_1^{-1}u) = 0$, for all $u \in U_{w_1}$, and

$$\mathbf{A}(\sigma, w_1) f(1) = 0. \qquad \square$$

Theorem 4.3. *Let (σ, V) be an irreducible admissible representation of M and $w_1, w_2, \ldots, w_r \in W(\sigma)$. Suppose that $\mathbf{A}(\nu, \sigma, w_i)$ is holomorphic at $\nu = 0$, for $i = 1, \ldots, r$. Then $T_{w_i}\mathbf{A}(\sigma, w_i)$, $i = 1, \ldots, r$, are linearly independent.*

Proof. Let

$$c_1 T_{w_1}\mathbf{A}(\sigma, w_1) + \cdots + c_r T_{w_2}\mathbf{A}(\sigma, w_r) = 0,$$

where $c_1, \ldots, c_r \in \mathbb{C}$. If $r = 1$, the statement is trivial. Suppose $r > 1$. Let w_i be a maximal element in the set $\{w_1, \ldots, w_r\}$. Then $w_i \neq 1$. According to Lemma 4.2, there exists a function $f_i \in i_{G,M}(V)$ such that

$$\mathbf{A}(\sigma, w_i) f_i(1) \neq 0 \quad \text{and} \quad \mathbf{A}(\sigma, w_j) f_i(1) = 0,$$

for $j \neq i$. Then

$$\bigl(c_1 T_{w_1}\mathbf{A}(\sigma, w_1) + \cdots + c_r T_{w_r}\mathbf{A}(\sigma, w_r)\bigr) f_i(1) = 0$$

implies $c_i = 0$. Repeating the arguments above $r - 1$ times, we prove $c_1 = c_2 = \cdots = c_r = 0$. $\square$

5. The Aubert involution and *R*-groups

Let D_M be the Aubert duality operator [3]. If σ is an irreducible representation of M, we denote by $\hat{\sigma}$ the representation $\pm D_M(\sigma)$, taking the sign $+$ or $-$ so that $\hat{\sigma}$ is a positive element in the Grothendieck group. We call $\hat{\sigma}$ the Aubert involution of σ. In this section and Section 6, we use results from [4], which are done under the hypothesis that the cocycle η is trivial, so we need the same hypothesis that $\eta = 1$, for both σ and $\hat{\sigma}$.

Theorem 5.1. *Let M be a standard Levi subgroup of G. Suppose that σ is an irreducible square integrable representation of M such that its Aubert involution $\hat{\sigma}$ is unitary. Let R be the R-group for $i_{G,M}(\sigma)$. Then $i_{G,M}(\hat{\sigma})$ has the same R-group in the following sense*:

the set of normalized standard intertwining operators

$$\{A(\hat{\sigma}, r) \mid r \in R\}$$

is a basis for the commuting algebra

$$C(\hat{\sigma}) = \mathrm{Hom}_G\big(i_{G,M}(\hat{\sigma}), i_{G,M}(\hat{\sigma})\big).$$

Lemma 5.2. *Under the assumptions of Theorem* 5.1*, the standard intertwining operators*

$$\{\mathbf{A}(\nu, \hat{\sigma}, r) \mid r \in R\}$$

are holomorphic at $\nu = 0$.

Proof. Take $w \in R$. Then $\mathbf{A}(\nu, \sigma, w)$ is holomorphic at $\nu = 0$. According to [20, Lemma 2.1.2] and [4, Corollary 6.3], there exists a family of subsets $\Theta_1, \dots, \Theta_{n+1} \subset \Delta$ such that

(1) $\Theta_1 = \Theta$ and $\Theta_{n+1} = \Theta'$;
(2) fix $1 \leqslant i \leqslant n$; then there exists a root $\alpha_i \in \Delta \backslash \Theta_i$ such that Θ_{i+1} is the conjugate of Θ_i in $\Omega_i = \Theta_i \cup \alpha_i$;
(3) set $w_i = w_{l,\Omega_i} w_{l,\Theta_i}$ in $W(\Theta_i, \Theta_{i+1})$ for $1 \leqslant i \leqslant n$; then

$$w = w_n \cdots w_1.$$

(4)
$$\mathbf{A}(\nu, \sigma, w) = \mathbf{A}(\nu_n, \sigma_n, w_n) \cdots \mathbf{A}(\nu_1, \sigma_1, w_1),$$
$$\mathbf{A}(\nu, \hat{\sigma}, w) = \mathbf{A}(\nu_n, \hat{\sigma}_n, w_n) \cdots \mathbf{A}(\nu_1, \hat{\sigma}_1, w_1),$$

where $\nu_1 = \nu$, $\sigma_1 = \sigma$, $\nu_i = w_{i-1}(\nu_{i-1})$, and $\sigma_i = w_{i-1}(\sigma_{i-1})$ for $2 \leqslant i \leqslant n$.

Moreover, $\mathbf{A}(\nu_i, \sigma_i, w_i)$ is holomorphic at $\nu_i = 0$, for all i. It follows from [4, Lemma 5.1] that

$$\mathbf{A}_G(\nu_i, \sigma_i, w_i) = i_{G,M_{\Omega_i}}\big(\mathbf{A}_{M_{\Omega_i}}(\nu_i, \sigma_i, w_i)\big),$$
$$\mathbf{A}_G(\nu_i, \hat{\sigma}_i, w_i) = i_{G,M_{\Omega_i}}\big(\mathbf{A}_{M_{\Omega_i}}(\nu_i, \hat{\sigma}_i, w_i)\big).$$

Now, [4, Lemma 7.1] tells us that $\mathbf{A}(\nu_i, \hat{\sigma}_i, w_i)$ is holomorphic at $\nu_i = 0$, for all i. Consequently, $\mathbf{A}(\nu, \hat{\sigma}, w)$ is holomorphic at $\nu = 0$. $\square$

Proof of Theorem 5.1. Lemma 5.2 and Theorem 4.3 imply that the set of standard intertwining operators $\{T_r\mathbf{A}(\hat{\sigma}, r) \mid r \in R\}$ is linearly independent. Therefore, $\{A(\hat{\sigma}, r) \mid r \in R\}$ is linearly independent. It is well-known [15,16] that

$$\left\{A(\sigma, r) \,\middle|\, r \in R\right\}$$

is a basis for the commuting algebra $C(\sigma)$. According to [4, Corollary 3.4],

$$C(\sigma) \cong C(\hat{\sigma})$$

and therefore $\dim C(\sigma) = \dim C(\hat{\sigma})$. This implies that

$$\left\{A(\hat{\sigma}, r) \,\middle|\, r \in R\right\}$$

is a basis for $C(\hat{\sigma})$. □

Other properties of R-groups are also inherited by $\hat{\sigma}$ [12].

Proposition 5.3. *Under the assumptions of Theorem* 5.1,

(1) *The inequivalent irreducible components $\hat{\pi}_i$ of $i_{G,M}(\hat{\sigma})$ are parametrized by irreducible representations $\rho_i = \rho(\hat{\pi}_i)$ of R.*
(2) *The multiplicity with which a component $\hat{\pi}_i = \hat{\pi}(\rho_i)$ occurs in $i_{G,M}(\hat{\sigma})$ is equal to the dimension of the representation ρ_i of R which parametrizes it.*

Proof. By [13, Theorem 2.4], (1) and (2) are satisfied for σ. Now (1) and (2) for $\hat{\sigma}$ follow directly, because $i_{G,M}(\sigma)$ and $i_{G,M}(\hat{\sigma})$ have the same number of irreducible components with same multiplicities and the correspondence between irreducible components is given by

$$\pi \leftrightarrow \hat{\pi}. \qquad \square$$

6. Normalized operators

In this section, we consider an irreducible unitary representation σ with the R-group R so that $\mathbb{C}[R] \cong \mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma))$. We describe explicitly the action of normalized operators $A(\sigma, r)$, $r \in R$, on irreducible subspaces of $i_{G,M}(\sigma)$ (see relation (17)) and express the result using a trace formulation (Theorem 6.1). Theorem 6.1 is closely related to [1, Conjecture 7.1].

We follow an outline written by the referee (cf. [13, pp. 38–40]). The multiplicative properties of normalized intertwining operators imply the map

$$r \mapsto A(\sigma, r)$$

gives an equivalence

$$\mathbb{C}[R] \cong \mathrm{Hom}_G\big(i_{G,M}(\sigma), i_{G,M}(\sigma)\big)$$

as R-modules ($r \in R$ acts on $\mathbb{C}[R]$ by left multiplication, on $\mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma))$ by composition on the left with $A(\sigma, r)$). We analyze $\mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma))$ and the action of intertwining operators by first decomposing the R-module $\mathbb{C}[R]$. That is, we decompose the left-regular representation of the finite group R [19, Section 2.4].

The correspondence between irreducible representations of R and components of $i_{G,M}(\sigma)$ may be established as in [13,14]. Let ρ be an irreducible representation of R. Denote by θ_ρ the character of ρ. Let $P_\rho \in \mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma))$ be defined by

$$P_\rho = \frac{\dim\rho}{|R|}\sum_r \overline{\theta_\rho(r)}A(\sigma, r) = \frac{\dim\rho}{|R|}\sum_r \theta_{\bar{\rho}}(r)A(\sigma, r).$$

The same argument as in [13] implies the P_ρ are the orthogonal projections onto the isotypic subspaces of $i_{G,M}(\sigma)$. Let

$$V_\rho = \mathrm{Im}(P_\rho).$$

To the representation ρ of R, one associates the irreducible representation $\pi(\rho)$, where V_ρ is the $\pi(\rho)$-isotypic subspace. The element of $\mathbb{C}[R]$ corresponding to P_ρ is

$$e_\rho = \frac{\dim\rho}{|R|}\sum_r \theta_{\bar{\rho}}(r)r.$$

We note that the ρ-isotypic subspace R_ρ of $\mathbb{C}[R]$ is

$$R_\rho = e_\rho\mathbb{C}[R] = \mathbb{C}[R]e_\rho.$$

This subspace contains $\dim\rho$ copies of ρ.

We now consider the left-regular representation and matrix coefficients. Let λ denote the left-regular representation of R on $\mathbb{C}[R]$. Let U be the space for ρ. Without loss of generality, we may assume this realization of ρ is unitary. For $u, u' \in U$, let $\rho_{u,u'}$ be the matrix coefficient for ρ given by

$$\rho_{u,u'}(r) = \langle \rho(r)u, u'\rangle.$$

One then has

$$\lambda(r)\rho_{u,u'} = \rho_{u,\rho(r)u'}.$$

We note that R_ρ may be identified with the space of matrix coefficients of ρ. As an element of $\mathbb{C}[R]$,

$$\rho_{u,v} = \sum_r \langle \rho(r)u, v\rangle r. \tag{12}$$

Choose an orthonormal basis $u_1, \dots, u_k$ of U. It follows that if

$$R^i_\rho = \operatorname{span}\{\rho_{u_i,u_1}, \dots, \rho_{u_i,u_k}\},$$

then

$$R_\rho = \bigoplus_i R^i_\rho$$

is a decomposition of R_ρ into left ideals.

Let $u_1, u_2, v_1, v_2 \in U$. Recall that Schur orthogonality says

$$\sum_r \langle \rho(r)u_1, u_2\rangle \overline{\langle \rho(r)v_1, v_2\rangle} = \frac{|R|}{\dim\rho} \langle u_1, v_1\rangle \overline{\langle u_2, v_2\rangle}. \tag{13}$$

It follows from (12) that

$$\rho_{u_1,u_2}\rho_{v_1,v_2} = \sum_{r,s} \langle \rho(r)u_1, u_2\rangle \langle \rho(s)u_1, u_2\rangle rs.$$

By substituting $t = rs$, we may transform the sum into

$$\sum_t \left(\sum_r \langle \rho(r)u_1, u_2\rangle \overline{\langle \rho(r)v_2, \rho(t)v_1\rangle} \right) t,$$

which is, according to (13), equal to

$$\sum_t \frac{|R|}{\dim\rho} \langle u_1, v_2\rangle \overline{\langle u_2, \rho(t)v_1\rangle} t = \frac{|R|\langle u_1, v_2\rangle}{\dim\rho} \rho_{v_1,u_2}.$$

Therefore,

$$\rho_{u_1,u_2}\rho_{v_1,v_2} = \frac{|R|}{\dim\rho} \langle u_1, v_2\rangle \rho_{v_1,u_2}. \tag{14}$$

It follows that projection onto R^i_ρ is given by right multiplication by the idempotent

$$e_{\rho,i} = \frac{\dim\rho}{|R|} \overline{\rho}_{u_i,u_i}.$$

That is, $R^i_\rho = R_\rho e_{\rho,i} = \mathbb{C}[R]e_{\rho,i}$. We can further decompose R^i_ρ (as a vector space only) using left multiplication

$$R^i_\rho = \mathbb{C}[R]e_{\rho,i} = e_{\rho,1}\mathbb{C}[R]e_{\rho,i} \oplus \cdots \oplus e_{\rho,k}\mathbb{C}[R]e_{\rho,i}.$$

(The subspaces $e_{\rho,j}\mathbb{C}[R]e_{\rho,i}$ will have a nice interpretation under the isomorphism $\mathbb{C}[R] \cong \mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma))$—see below.)

We now transfer this information to the other side of the isomorphism. Note that under the isomorphism $\mathbb{C}[R] \cong \mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma))$, $R_\rho \cong \mathrm{Hom}_G(V_\rho, V_\rho)$ (suitably interpreted). Now, let P_ρ^i denote the element of $\mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma))$ corresponding to $e_{\rho,i}$ under the isomorphism $\mathbb{C}[R] \cong \mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma))$. Then,

$$V_\rho = \bigoplus_i V_\rho^i,$$

with $V_\rho^i = \mathrm{Im}(P_\rho^i)$, is a decomposition of V_ρ into invariant subspaces. To interpret this in terms of the action of the normalized intertwining operators, observe that under the isomorphism $\mathbb{C}[R] \cong \mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma))$, we have

$$\mathrm{Hom}_G\big(V_\rho^i, V_\rho^j\big) \cong P_\rho^j \,\mathrm{Hom}_G\big(i_{G,M}(\sigma), i_{G,M}(\sigma)\big) P_\rho^i \longleftrightarrow e_{\rho,j}\mathbb{C}[R]e_{\rho,i}.$$

For $r \in R$, the correspondence is

$$P_\rho^j A(\sigma, r) P_\rho^i \longleftrightarrow e_{\rho,j} r e_{\rho,i}. \tag{15}$$

To compute the right-hand side of (15), we first find, using (12),

$$\overline{\rho}_{u,v} r = \overline{\rho}_{\rho(r^{-1})u,v}.$$

According to (14), we have

$$\begin{aligned} e_{\rho,j} r e_{\rho,i} &= \frac{(\dim\rho)^2}{|R|^2} \overline{\rho}_{\rho(r^{-1})u_j,u_j} \overline{\rho}_{u_i,u_i} = \frac{(\dim\rho)}{|R|} \overline{\langle \rho(r^{-1})u_j, u_i\rangle} \overline{\rho}_{u_i,u_j} \\ &= \rho_{u_i,u_j}(r) \frac{\dim\rho}{|R|} \overline{\rho}_{u_i,u_j}. \end{aligned}$$

For the left-hand side of (15), Schur's lemma implies $\mathrm{Hom}_G(V_\rho^i, V_\rho^j) \cong \mathbb{C} I_{i,j}$. Here, we may take $I_{i,j} \in \mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma))$ corresponding to $|R|^{-1}(\dim\rho)\overline{\rho}_{u_i,u_j} \in \mathbb{C}[R]$ under the isomorphism $\mathbb{C}[R] \cong \mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma))$. In particular, we have

$$P_\rho^j A(\sigma, r) P_\rho^i = \rho_{u_i,u_j}(r) I_{i,j}. \tag{16}$$

We do this process for each irreducible representation ρ of R, that is, we fix an orthonormal basis $u_1^\rho, \ldots, u_{k_\rho}^\rho$ of the space of ρ and define projections P_ρ^i. Then (16) holds. Further, if ρ and ρ' are two inequivalent representations of R, we have

$$P_{\rho'}^j A(\sigma, r) P_\rho^i = 0.$$

Notice that

$$\mathrm{id} = \sum_{\rho} \sum_{i} P_{\rho}^{i}.$$

Therefore,

$$A(\sigma, r) = \sum_{\rho'} \sum_{j} P_{\rho'}^{j} A(\sigma, r) \sum_{\rho} \sum_{i} P_{\rho}^{i} = \sum_{\rho} \sum_{i,j} P_{\rho}^{j} A(\sigma, r) P_{\rho}^{i},$$

and we have

$$A(\sigma, r) = \sum_{\rho} \sum_{i,j} \rho_{u_i, u_j}(r) I_{i,j}. \tag{17}$$

This describes explicitly the action of the normalized intertwining operator $A(\sigma, r)$ on irreducible subspaces. The relation (17), however, depends on the choice of bases or, equivalently, the choice of irreducible subspaces.

We will express the result using a trace formulation (cf. [13, Theorem 2.7], and [1, Conjecture 7.1]). For $f \in C_c^{\infty}(G)$ and for a representation π of G, we define

$$I(f) = \int_G f(g) i_{G,M}(\sigma)(g)\,\mathrm{d}g \quad \text{and} \quad \pi(f) = \int_G f(g)\pi(g)\,\mathrm{d}g.$$

Let ρ be an irreducible representation of R and let $\pi = \pi_{\rho}$ be an equivalence class of irreducible components of $i_{G,M}(\sigma)$ associated to ρ. We define

$$\langle r, \pi \rangle = \operatorname{trace} \rho(r), \quad \text{for } r \in R.$$

Theorem 6.1. *Let σ be an irreducible unitary representation of M. Suppose that $\mathbb{C}[R] \cong \operatorname{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\sigma))$. Then, for $r \in R$ and $f \in C_c^{\infty}(G)$,*

$$\operatorname{trace}\bigl(A(\sigma, r) I(f)\bigr) = \sum_{\pi} \langle r, \pi \rangle \operatorname{trace} \pi(f),$$

where the sum runs over all equivalence classes π of irreducible components of $i_{G,M}(\sigma)$.

Proof. According to (17), we have

$$\begin{aligned}
\operatorname{trace}\bigl(A(\sigma, r) I(f)\bigr) &= \operatorname{trace}\left(\sum_{\rho} \sum_{i,j} \rho_{u_i, u_j}(r) I_{i,j} I(f)\right) \\
&= \sum_{\rho} \sum_{i} \rho_{u_i, u_i}(r) \operatorname{trace} \pi_{\rho}(f) = \sum_{\rho} \operatorname{trace} \rho(r) \operatorname{trace} \pi_{\rho}(f) \\
&= \sum_{\pi} \langle r, \pi \rangle \operatorname{trace} \pi(f). \qquad \square
\end{aligned}$$

In particular, Theorem 6.1 applies to representations considered in Section 5.

Corollary 6.2. *Let σ be an irreducible square integrable representation of M. Suppose that the Aubert involution $\hat{\sigma}$ is unitary. Then,*

$$\operatorname{trace}\big(A(\hat{\sigma}, r)I(f)\big) = \sum_{\hat{\pi}} \langle r, \hat{\pi}\rangle \operatorname{trace}\hat{\pi}(f),$$

where the sum runs over all equivalence classes $\hat{\pi}$ of irreducible components of $i_{G,M}(\hat{\sigma})$.

Proof. Follows from Theorems 5.1 and 6.1. □

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